Estimation

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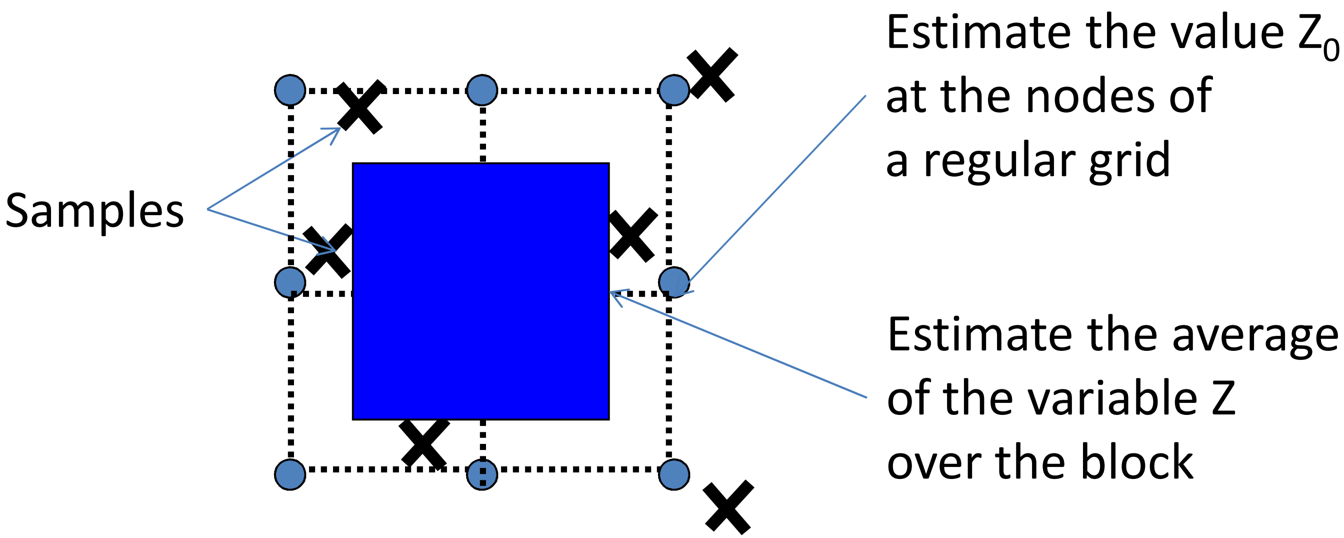
o What can be estimated ?

- Punctual estimation
- Block average estimation

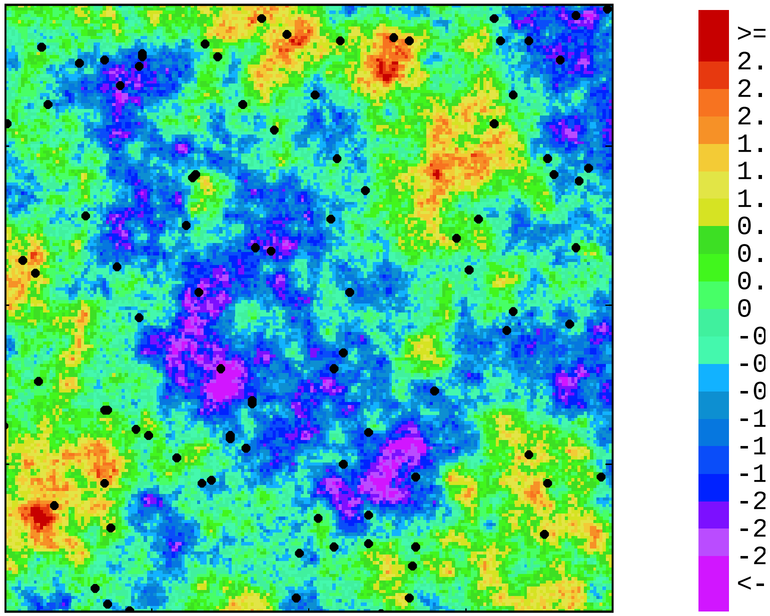


o Linear estimation techniques

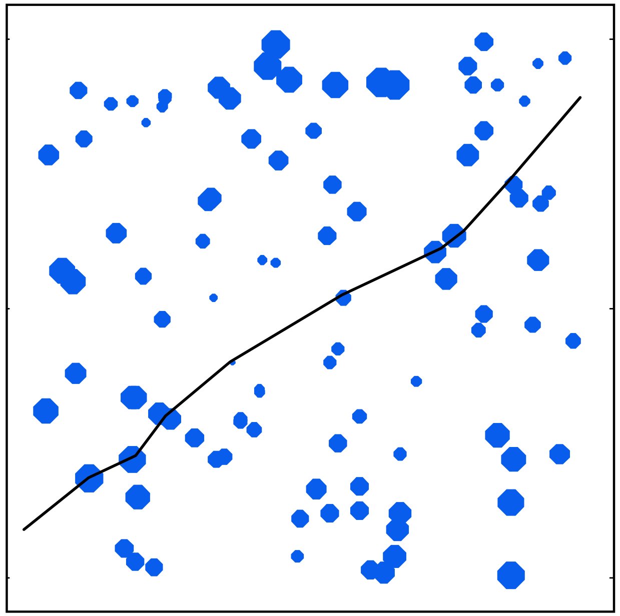
- Linear estimator:
  - Each estimation is obtained as a linear combination of the values measured at sample points
- Several linear interpolation techniques:
  - Moving average
  - Inverse distance (closest point)
  - Inverse distance
- Properties of the estimation:
  - Smoothness
  - Unbiasedness
  - Exact interpolation

o Illustration

- Exhaustive data set (reality)
- Irregular sampling used as data
- For each method, represent the estimation as a map and along a section



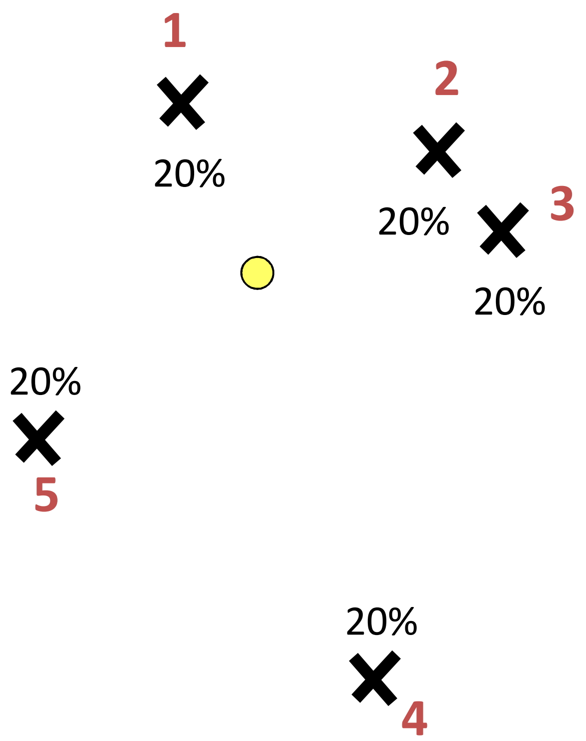
Reality

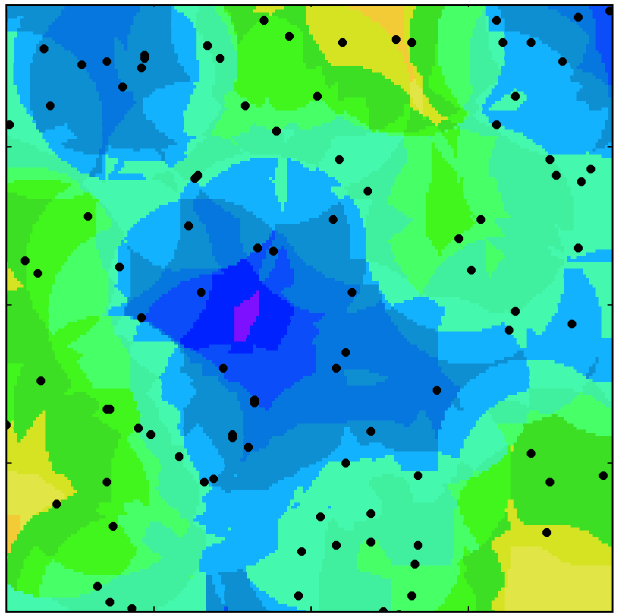


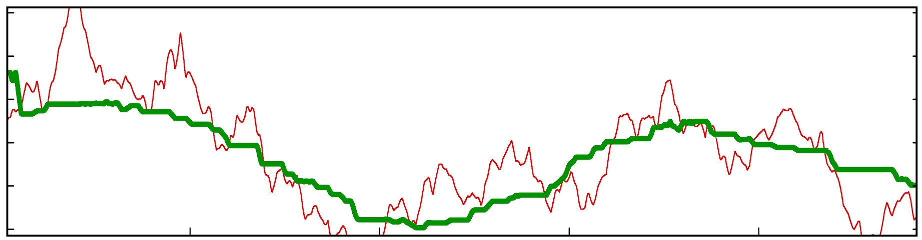
Data & Section

o Moving average

$$z^* = \frac{\sum z_i}{5}$$

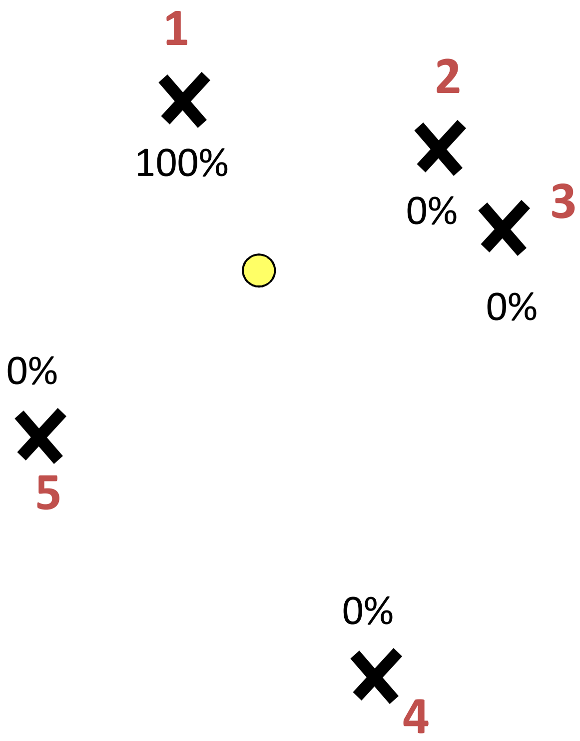


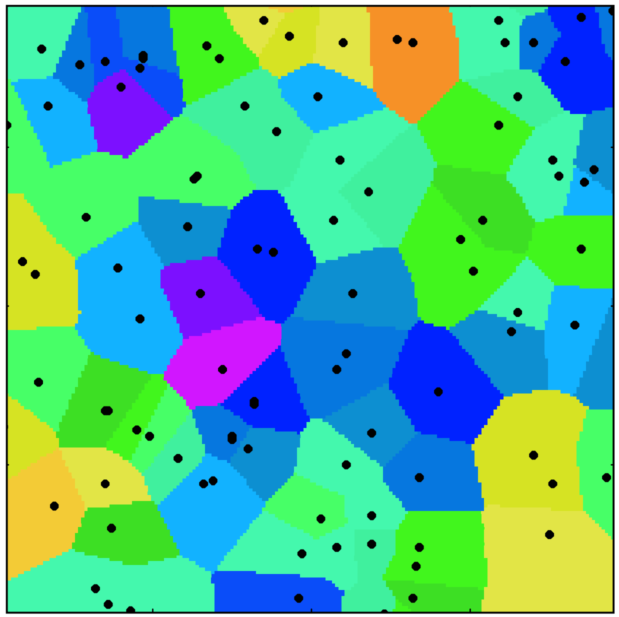


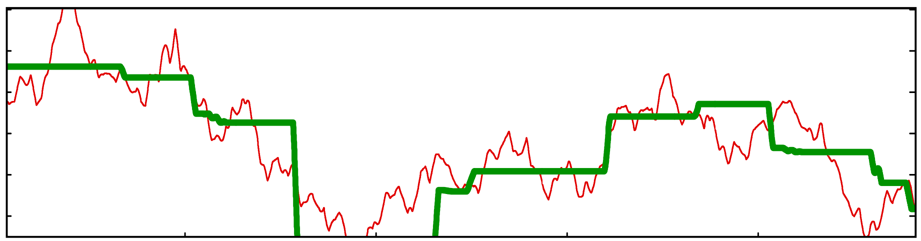


o Influence polygon – Nearest neighbor

$$z^{*} = z_{1}$$

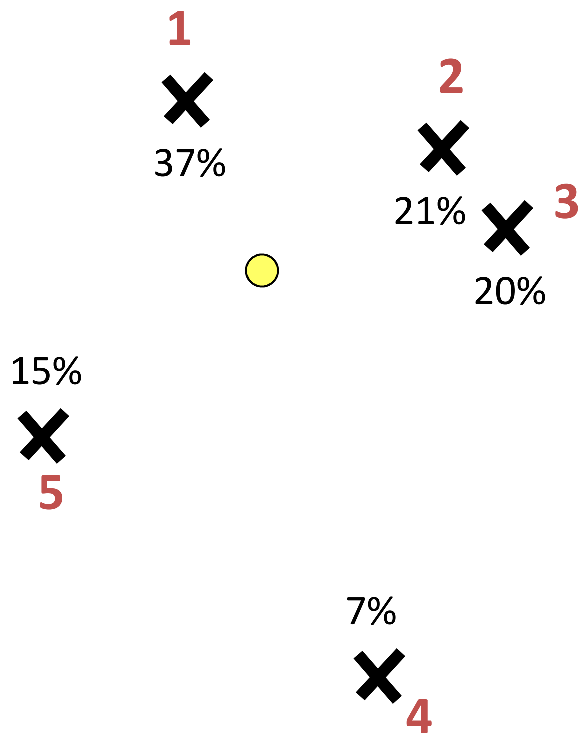


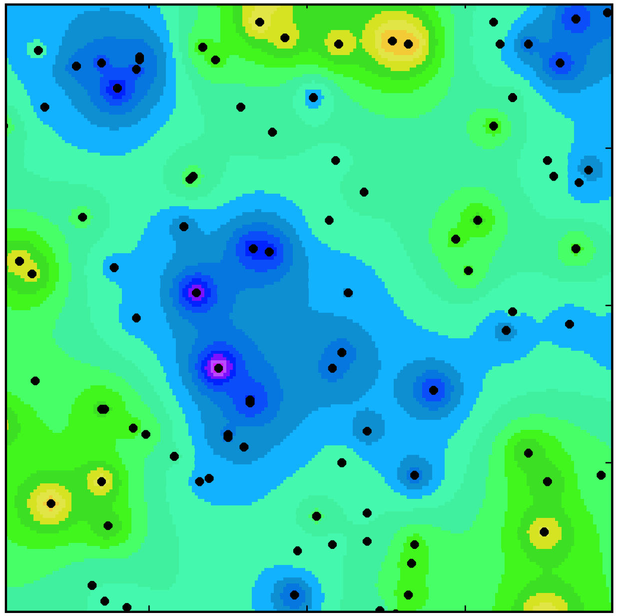


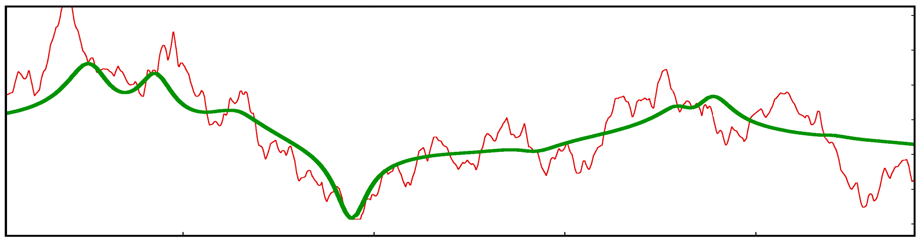


o Inverse distance

$$z^* = \frac{\sum z_i / d_i^2}{\sum 1 / d_i^2}$$







o What do we want to achieve ?

- Produce an estimation of the variable z at the target location, as a linear combination of the sample values:

$$z_0^*$$

- If the real unknown value is denoted:

$$z_0$$

- We want the estimation error:

$$\varepsilon = z_0 - z_0^*$$

  - To be zero on average
  - To be as small as possible (quadratic)
  - To take into account:
    - Distances (between samples and target, and among samples)
    - Spatial characteristics: continuity, smoothness, …

o Formalism – Random framework

- Reality is unknown, unique and complex
- It would be impossible to reproduce all the processes involved which have produced the sample values
- Hence the choice of probabilistic framework
- The regionalized variable is considered as a realization (outcome) of a random function:

$$z(x) = Z(x, \omega) = Z(x)$$

o Random Functions

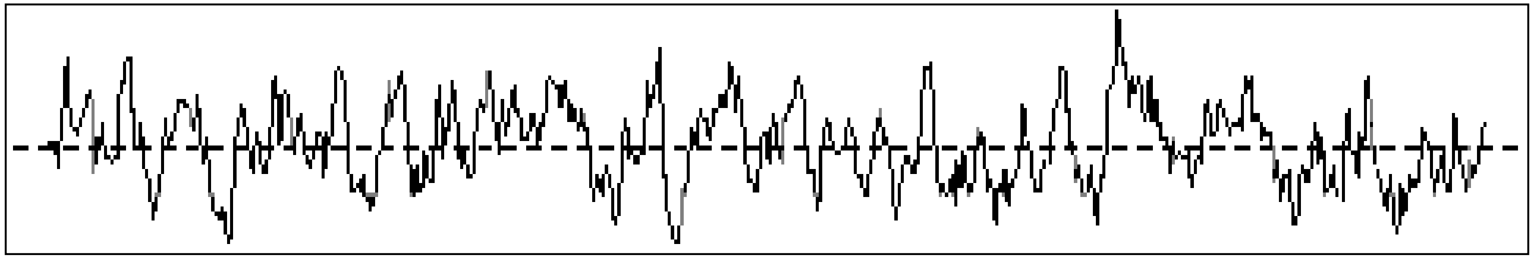
- Stationary (or order 2):
  - Mean:

$$E(Z) = m$$

  - Covariance :

$$Cov[Z(x), Z(x+h)] = E[Z(x)-m][Z(x+h)-m] = C(h)$$

  - Variance: $$Var[Z(x)] = C(0)$$



Example of a stationary Random Function (in 1-D)

o Random Functions

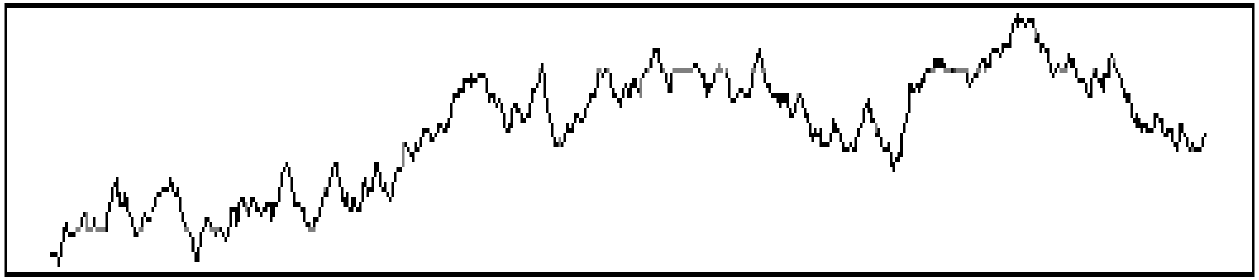
- Intrinsic (or order 0): increments are stationary
  - Mean:

$$E\left[Z(x+h)-Z(x)\right]=0$$

  - Variance → Variogram

$$\gamma(h)=\frac{1}{2}Var\left[Z(x+h)-Z(x)\right]=\frac{1}{2}E\left[Z(x+h)-Z(x)\right]^2$$

  - Stationary → Intrinsic



Example of an Intrinsic Random Function (in 1-D)

o Kriging

- Estimation of the variable Z at the target location:

$$Z_0^* = \sum \lambda_\alpha Z(x_\alpha)$$

- The estimation error:

$$\varepsilon = Z_0 - Z_0^*$$

  - Must have a zero expectation

$$\mathrm{E}(\varepsilon) = 0$$

  - And minimum variance:

$$Var(\varepsilon) \quad \text{minimum}$$

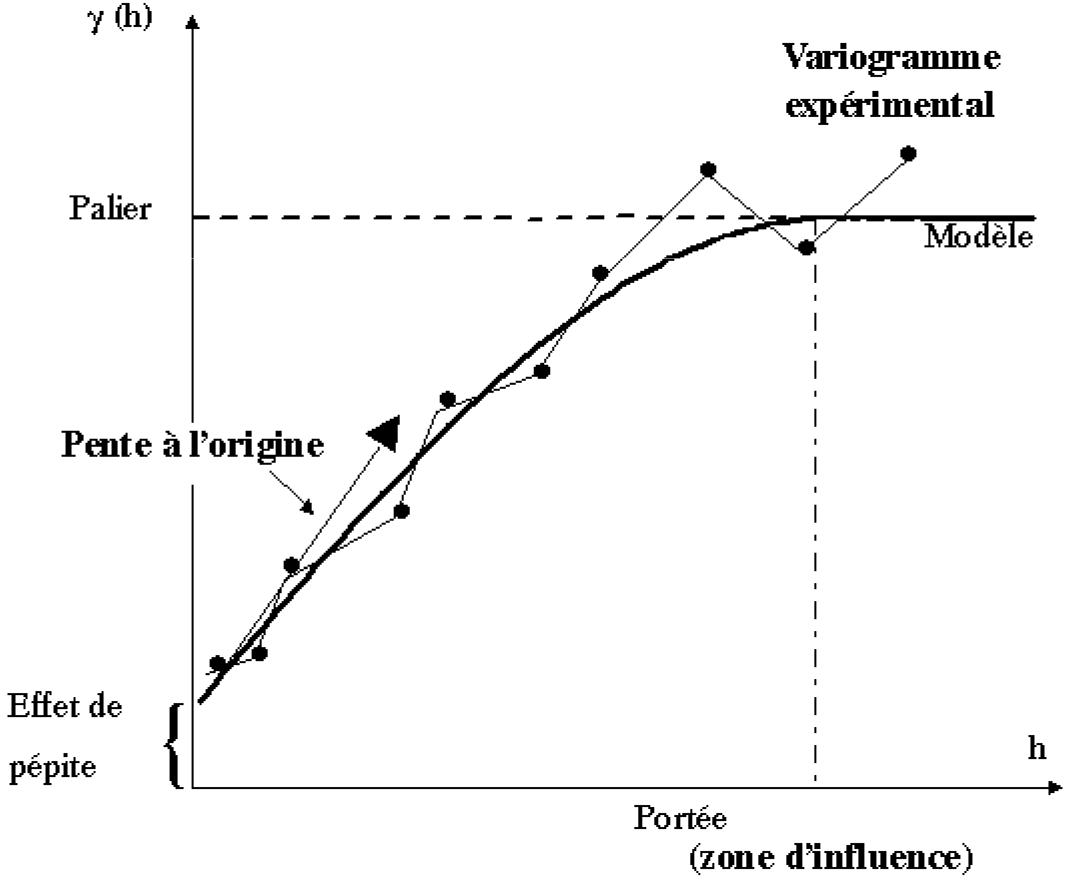
- This method is named **Kriging**

o Need for a Model

- Last constraint requires the calculation of the variance of a linear combination which must remain positive.
- This calls for the use of a valid model (positive definite property)

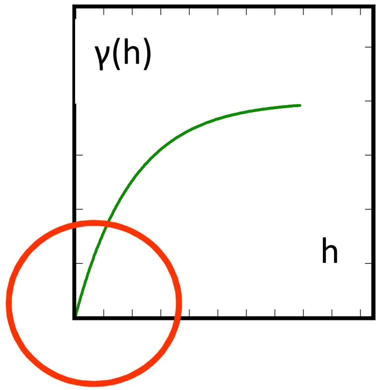
o General characteristics

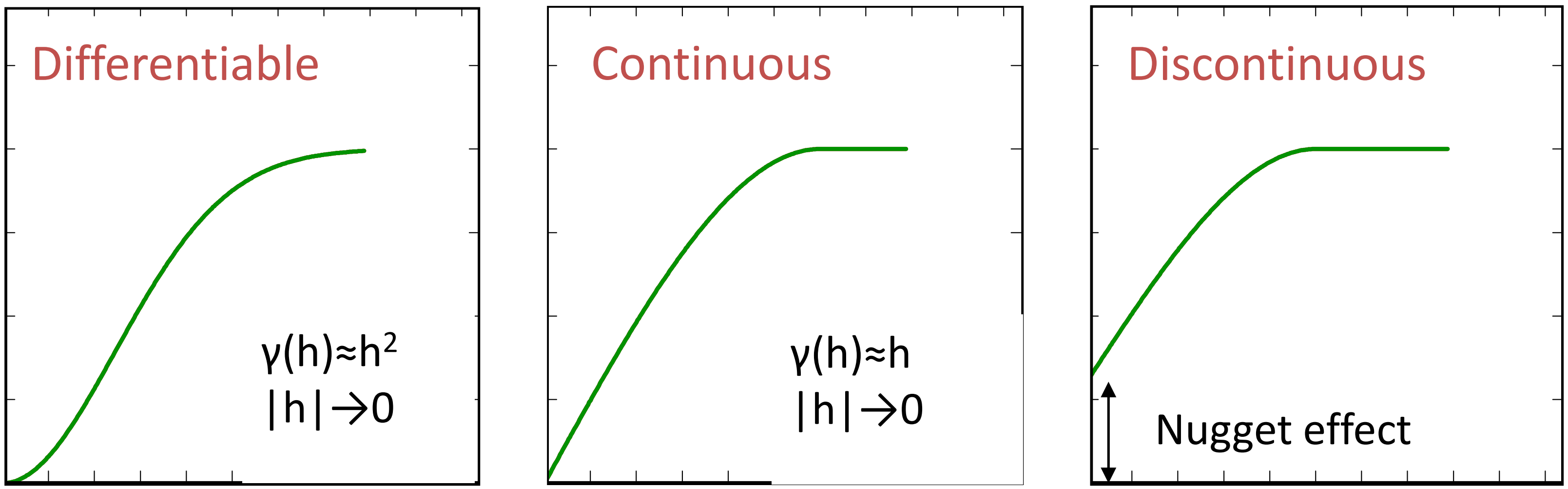
- Fitting the model on the experimental variograms (or covariances) calculated experimentally from the data



o General characteristics

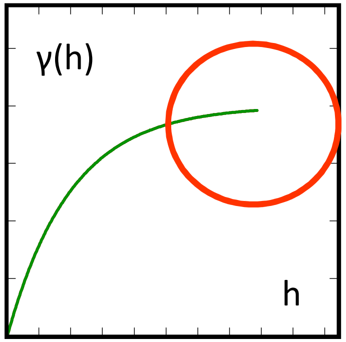
- Behavior at the origin describes the regularity of the variable

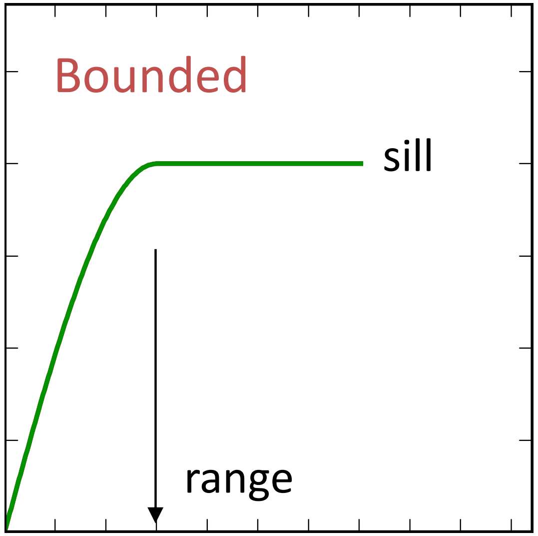


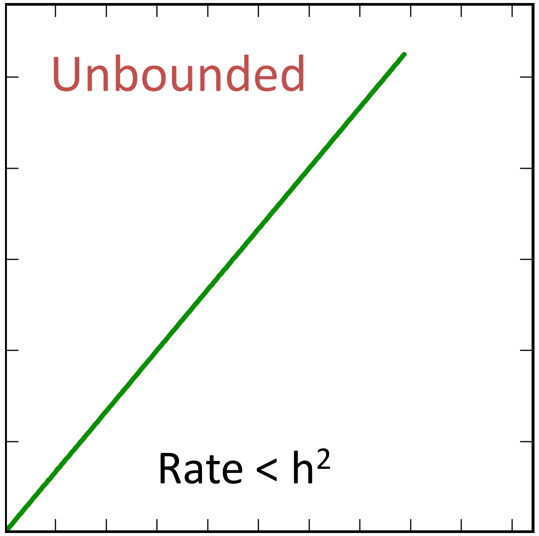


o General characteristics

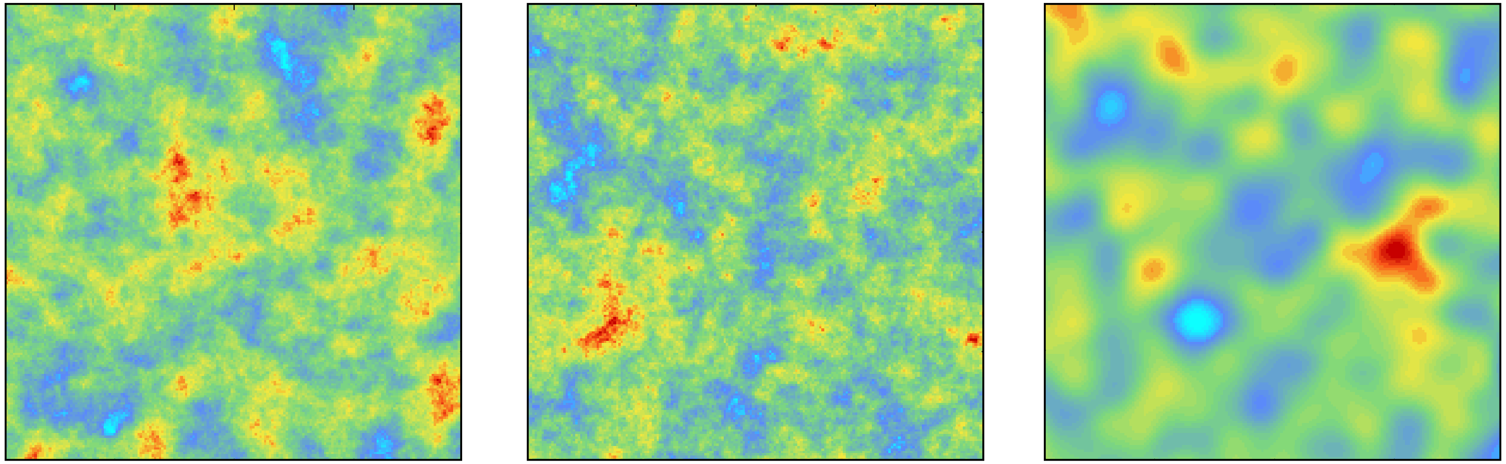
- Behavior at large distances



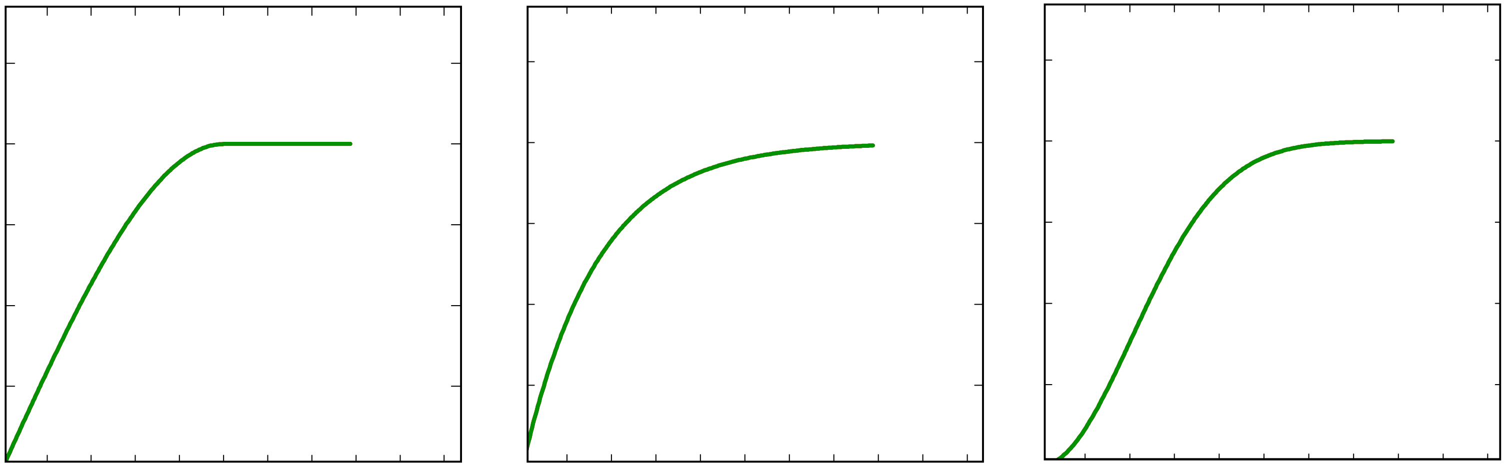




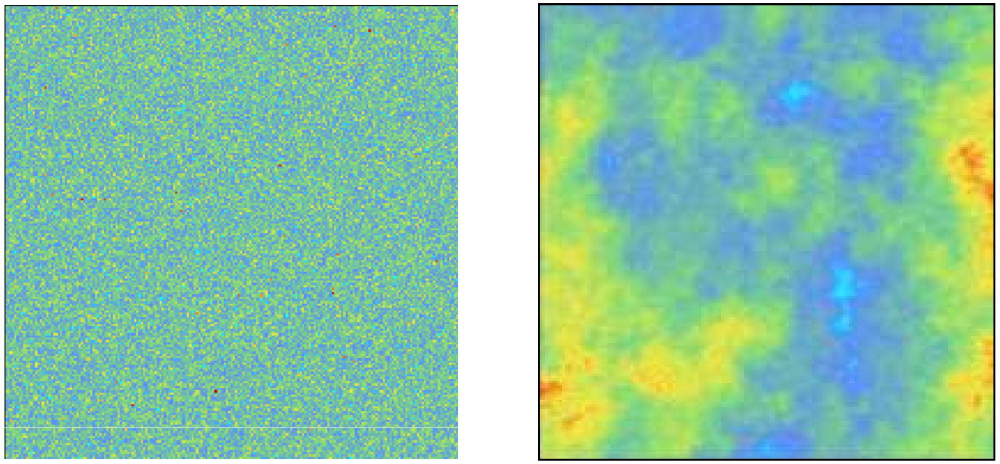
o Different structures



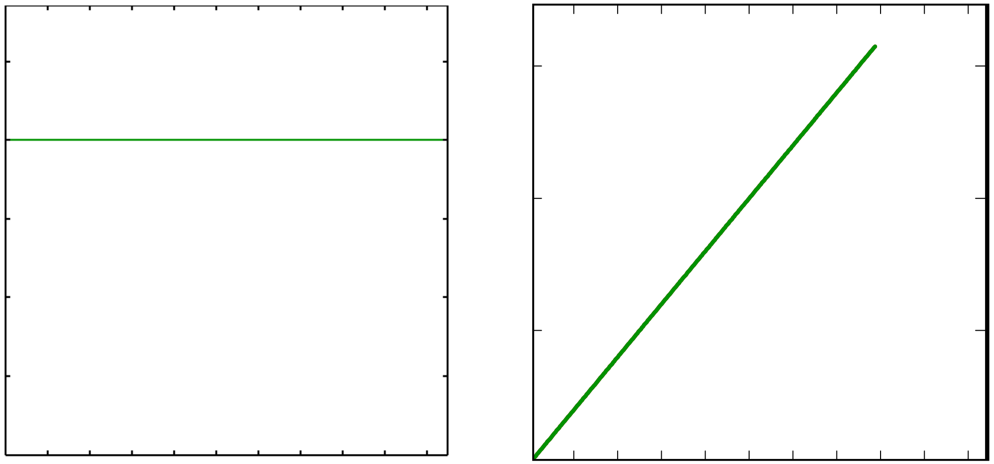
Spherical   Exponential   Gaussian



o Different structures



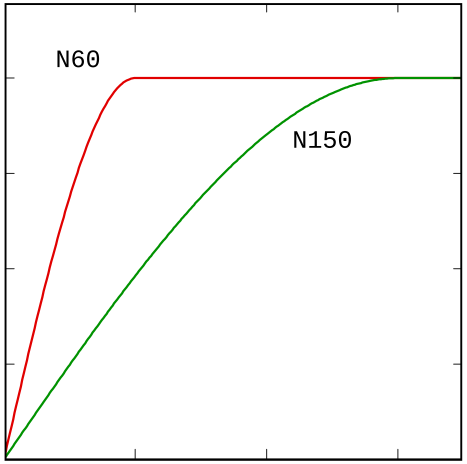
Nugget Effect | Linear

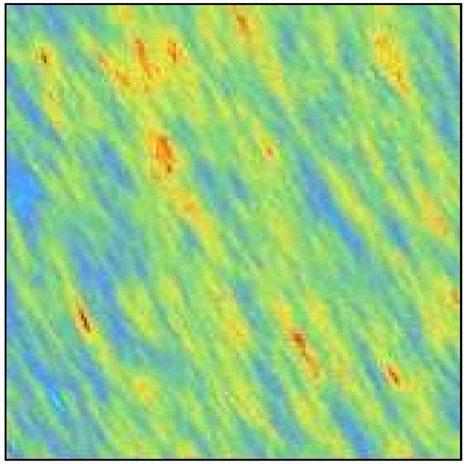


o Anisotropies

Geometrical

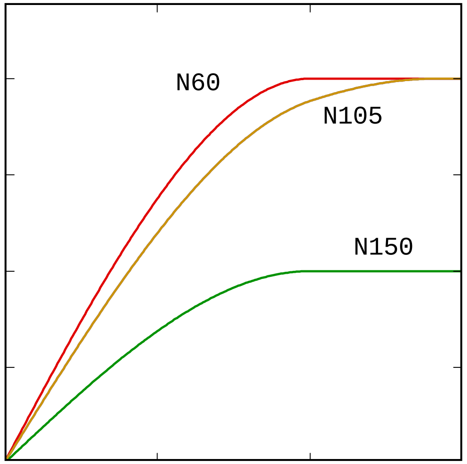
$$\gamma\left(\sqrt{\left(\frac{h_u}{a_u}\right)^2+\left(\frac{h_v}{a_v}\right)^2}\right)$$

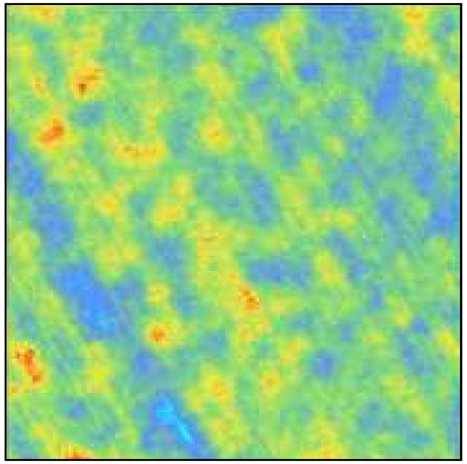




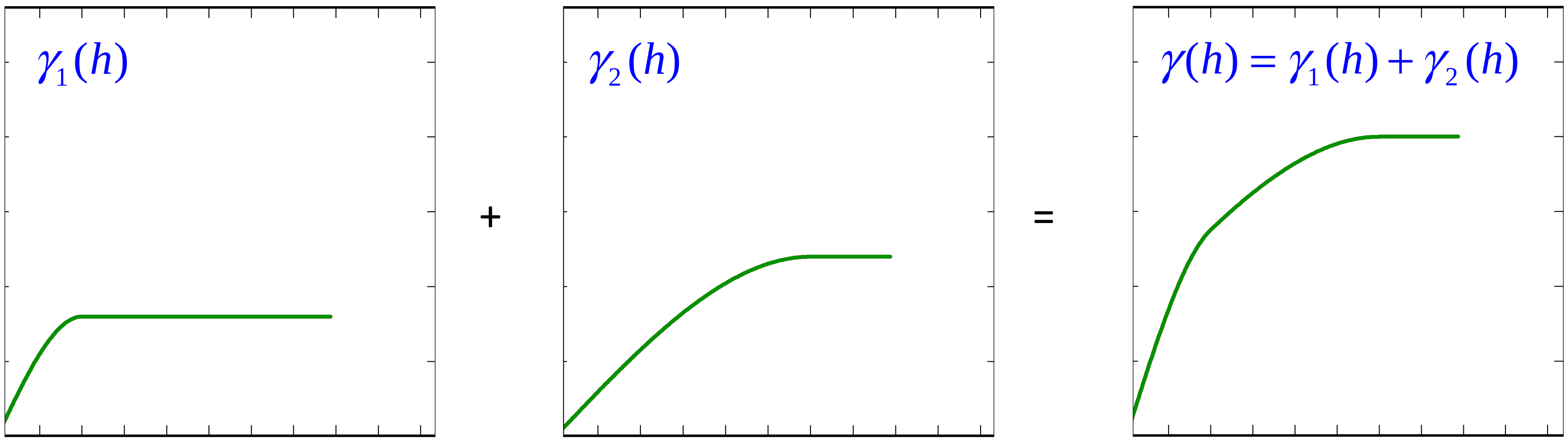
Zonal

$$\gamma(h_u,h_v)+\gamma(h_u)$$

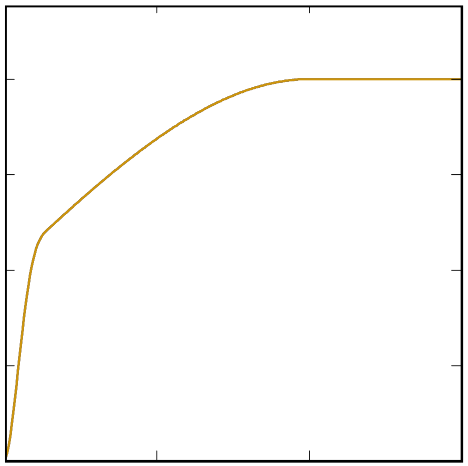


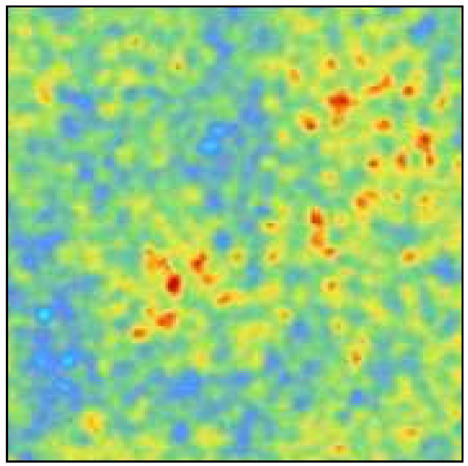


o Nesting structures



Example: Nested model:
- short range cubic
- spherical long range



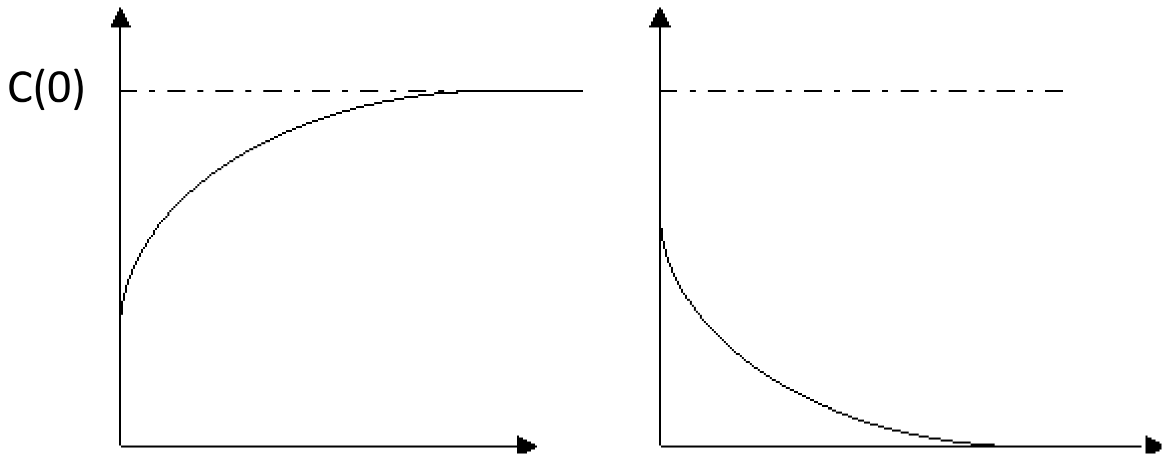


o Link between covariance and variogram

- A covariance is a (bounded) variogram
- An (unbounded) variogram cannot be a covariance

- When a covariance exists, the link between covariance and variogram is:

$$\gamma(h) = C(0) - C(h)$$



o Reminders

- Estimation of the variable Z at the target location:

$$Z_0^* = \sum \lambda_\alpha Z(x_\alpha)$$

- The estimation error:

$$\varepsilon = Z_0 - Z_0^*$$

  - Must have a zero expectation

$$\mathrm{E}(\varepsilon) = 0$$

  - And minimum variance:

$$Var(\varepsilon) \quad \text{minimum}$$

- This method is named **Kriging**

o Principle

- Z is a **stationary** Random Variable with a **constant known mean**:

$$m = E[Z]$$

- The estimation is obtained as a linear combination of data :

$$Z_0^* = \sum_\alpha \lambda_\alpha Z_\alpha + m\left(1 - \sum_\alpha \lambda_\alpha\right)$$

- where the Kriging weights are obtained as solution of the Kriging system:

$$\sum_\beta \lambda_\beta C_{\alpha\beta} = C_{\alpha 0}$$

- We also obtain the variance of the estimation error:

$$Var[\varepsilon] = C_{00} - \sum_\alpha \lambda_\alpha C_{\alpha 0}$$

o In matrix notation

- Kriging system (regular if no duplicate):

$$\left[C_{\alpha\beta}\right]\times[\lambda_\alpha]=[C_{\alpha 0}]$$

- Estimation:

$$Z_0^* = [Z_\alpha]^t \times [\lambda_\alpha] + m \times \left(1 - \sum_\alpha \lambda_\alpha\right)$$

- Variance of the estimation error:

$$Var(\varepsilon) = C_{00} - [\lambda_\alpha]^t \times [C_{\alpha 0}]$$

o Properties

- Kriging is a smoothed estimation

$$Var\left(Z_0^*\right) \leq Var\left(Z_0\right)$$

- Kriging is an exact interpolation: at data location, kriging estimate matches data value and estimation error is zero:

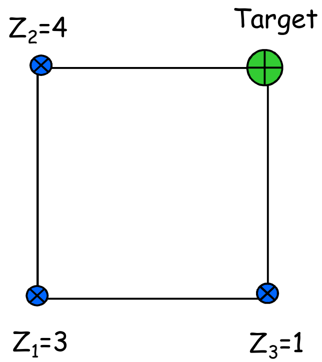
$$Z^*(x_\alpha) = Z_\alpha \quad \text{and} \quad Var\left(\varepsilon_\alpha\right) = 0$$

- Kriging weights do not depend on data values
- The estimation does not depend on the covariance sill
- The variance of estimation error is directly proportional to the covariance sill

o Exercise

Simple Kriging in the following setup:

- 3 Data and Target on a square pattern (mesh = 1m)
- Spherical covariance with range 1.25m and sill 2
- Known mean = 2



- Establish and solve the simple kriging system
- Derive the estimated value and the corresponding estimation variance

o Exercise

- Simple Kriging system:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \times \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} C_{10} \\ C_{20} \\ C_{30} \end{bmatrix}$$

$$C(0) = 2$$
$$C(1) = 0.112$$
$$C(\sqrt{2}) = 0$$

- Kriging weights:

$$\lambda_1 = -0.006$$
$$\lambda_2 = \lambda_3 = 0.056$$
$$\sum \lambda_i = 0.106$$

- Results:

$$Z^* = 2.050$$
$$\sigma^2 = 1.410$$

Ordinary Kriging

o Principle

- Z is a **stationary** Random Variable with a **constant unknown mean**:
- The estimation is obtained as a linear combination of data :

$$Z_0^* = \sum_\alpha \lambda_\alpha Z_\alpha$$

- where the Kriging weights are obtained as solution of the Kriging system:

$$\begin{cases} \sum_\beta \lambda_\beta C_{\alpha\beta} + \mu & = C_{\alpha 0} \\ \sum_\beta \lambda_\beta & = 1 \end{cases}$$

- We also obtain the variance of the estimation error:

$$Var(\varepsilon) = C_{00} - \sum_\alpha \lambda_\alpha C_{\alpha 0} - \mu$$

o In matrix notation

- Kriging system (regular if no duplicate):

$$\begin{bmatrix} C_{\alpha\beta} & 1 \\ 1^t & 0 \end{bmatrix} \times \begin{bmatrix} \lambda_\alpha \\ \mu \end{bmatrix} = \begin{bmatrix} C_{\alpha 0} \\ 1 \end{bmatrix}$$

- Estimation:

$$Z^* = \begin{bmatrix} Z_\alpha \\ 0 \end{bmatrix}^t \times \begin{bmatrix} \lambda_\alpha \\ \mu \end{bmatrix}$$

- Variance of the estimation error:

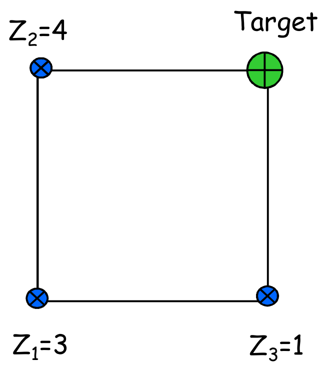
$$Var(\varepsilon) = C_{00} - \begin{bmatrix} \lambda_\alpha \\ \mu \end{bmatrix}^t \times \begin{bmatrix} C_{\alpha 0} \\ 1 \end{bmatrix}$$

- Can also be written replacing $C(h)$ by $-\gamma(h)$

o Exercise

Ordinary Kriging in the following setup:

- 3 Data and Target on a square pattern (mesh = 1m)
- Spherical covariance with range 1.25m and sill 2



- Establish and solve the ordinary kriging system
- Derive the estimated value and the corresponding estimation variance

o Exercise

- Ordinary Kriging system:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 1 \\ C_{21} & C_{22} & C_{23} & 1 \\ C_{31} & C_{32} & C_{3} & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \mu \end{bmatrix} = \begin{bmatrix} C_{10} \\ C_{20} \\ C_{30} \\ 1 \end{bmatrix}$$

$$C(0) = 2$$

$$C(1) = 0.112$$

$$C(\sqrt{2}) = 0$$

- Kriging weights:

$$\lambda_1 = 0.280$$

$$\lambda_2 = \lambda_3 = 0.360$$

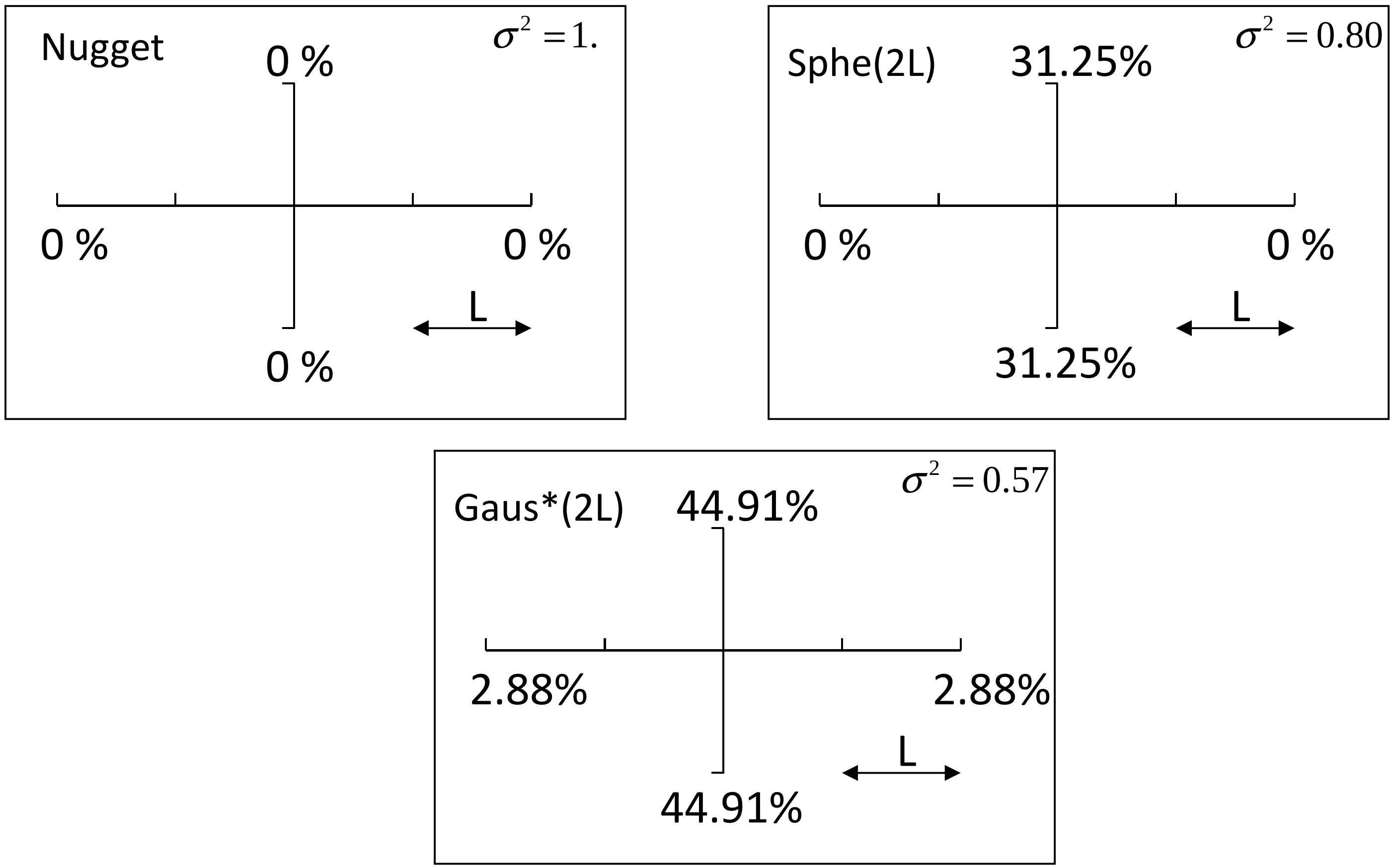
$$\mu = -0.600$$

- Results:

$$Z^* = 2.640$$

$$\sigma^2 = 1.600$$

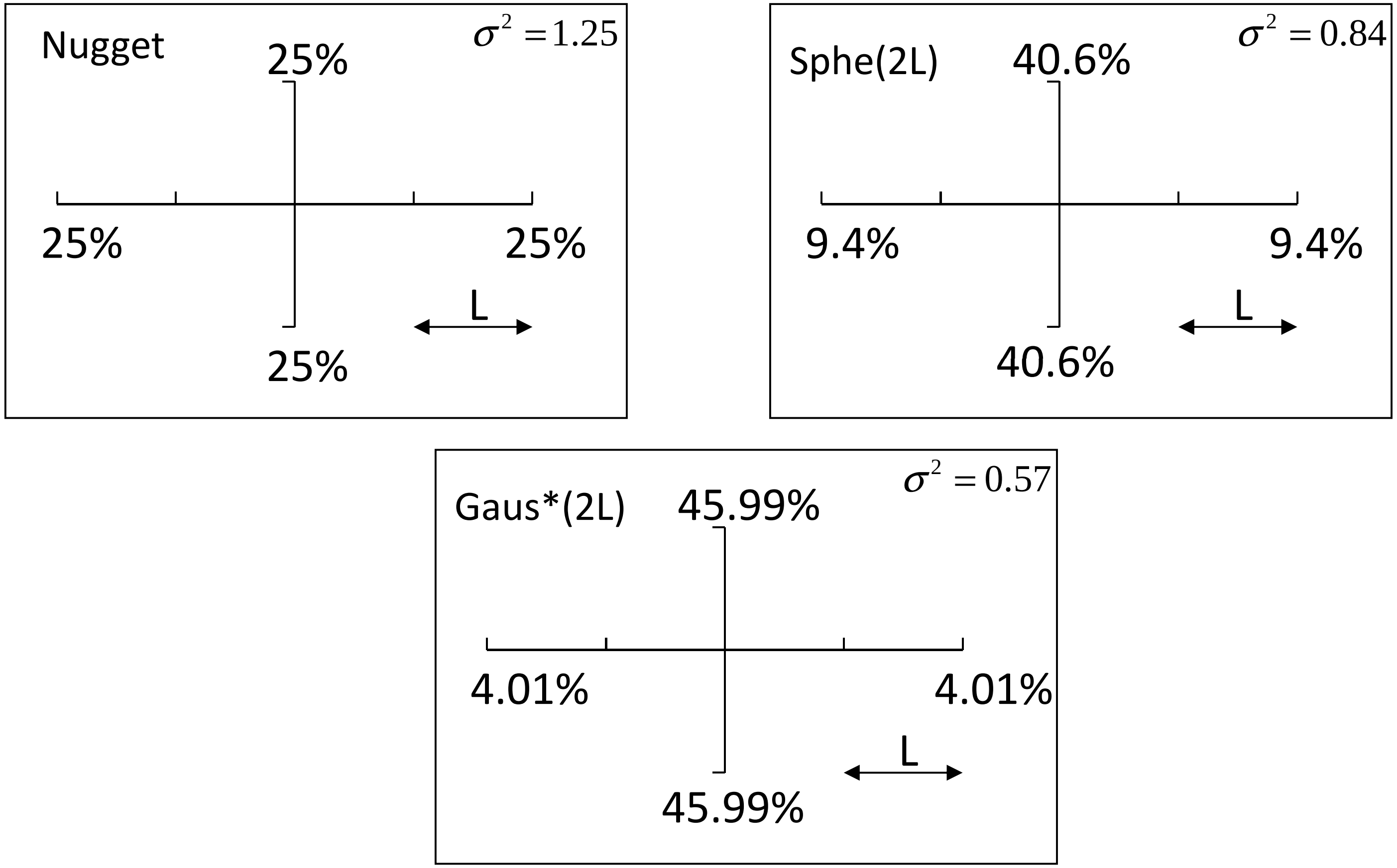
o Simple Kriging



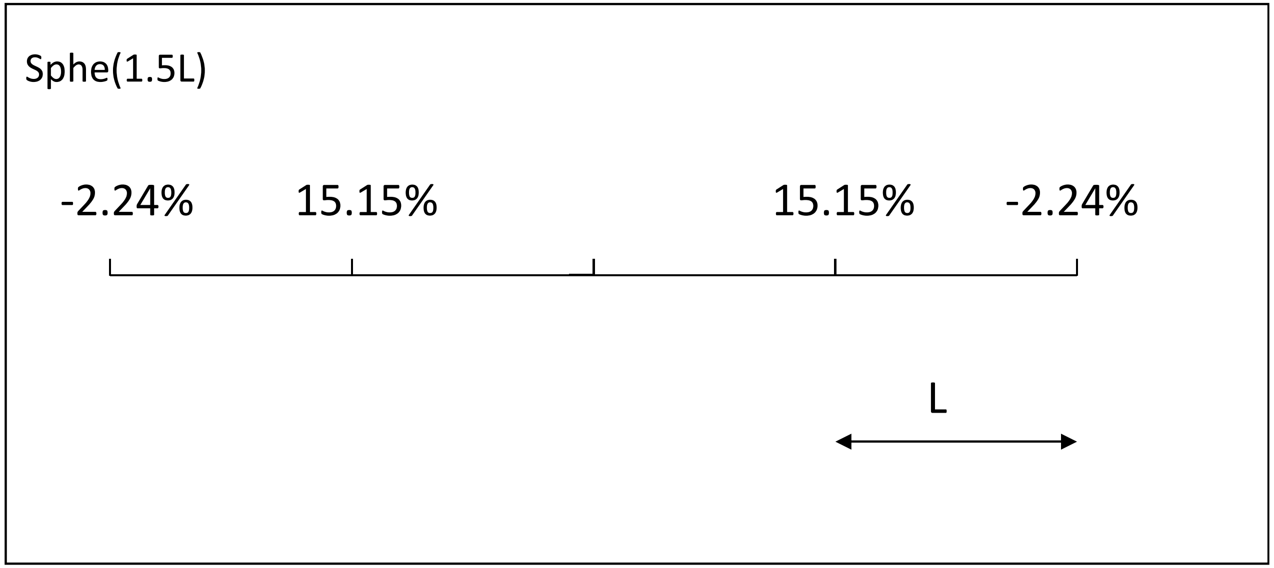
mean=0

Kriging weights

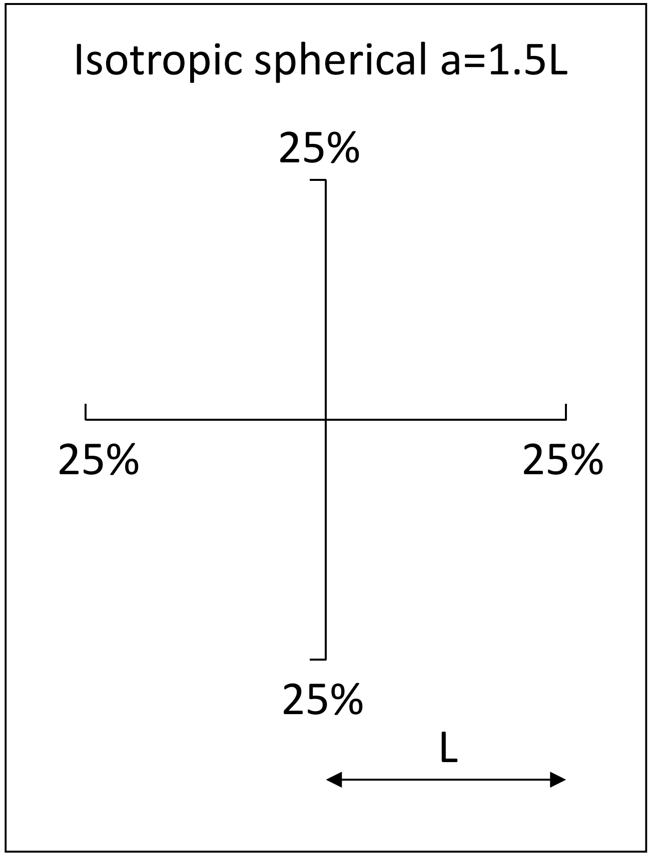
o Ordinary Kriging



o Relay in Simple Kriging



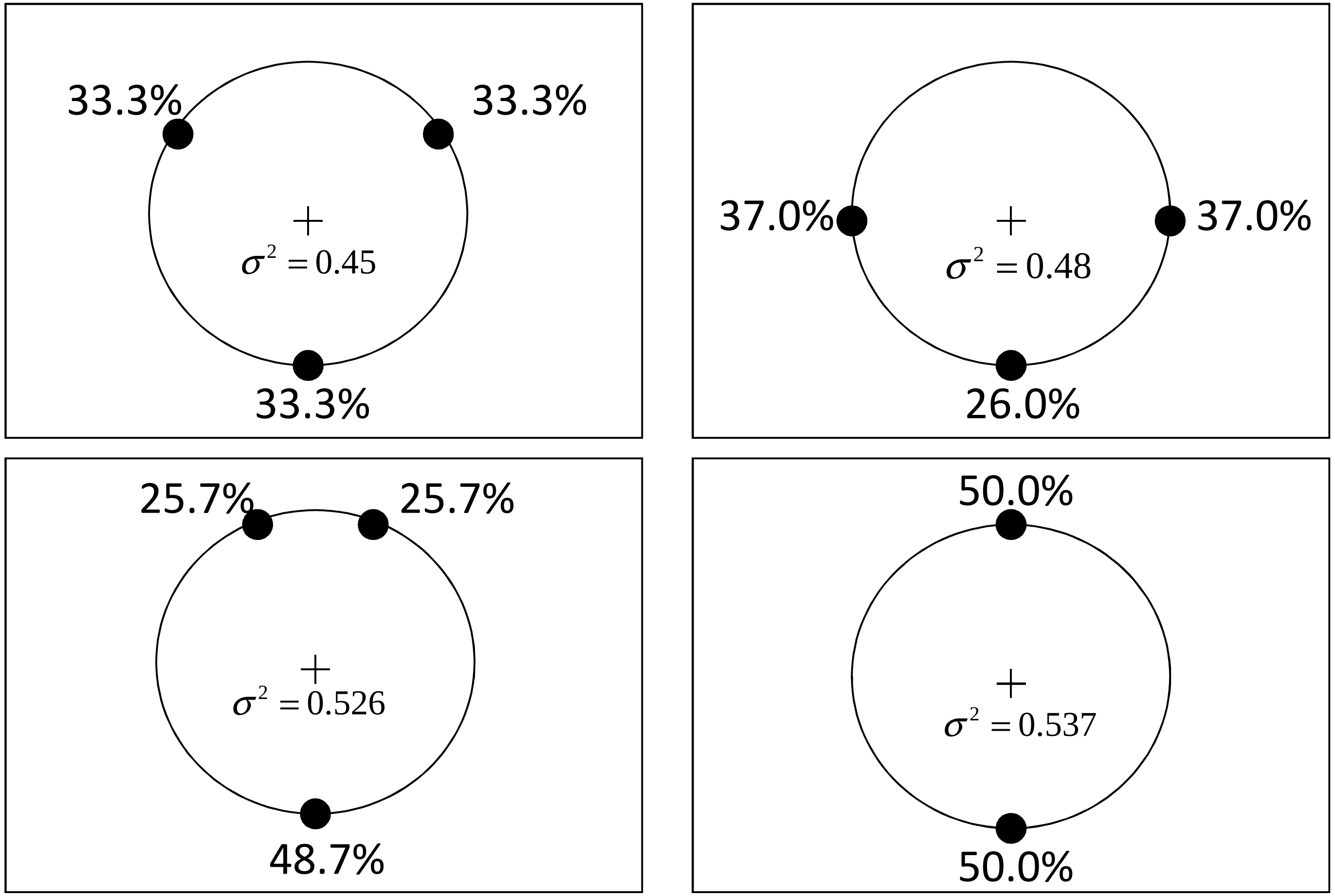
o Anisotropy



Kriging weights

o Declustering effect

Isotropic spherical with range > radius



o Principle

- At each data point:
  - Suppress the sample value
  - Estimate its value by Kriging
  - Compare real to estimated values

$$Z^*_{\alpha_0} = \sum_{\alpha \neq \alpha_0} \lambda_\alpha Z_\alpha$$

- Statistics on:
  - Error: $$\varepsilon_\alpha = Z^*_{\alpha_0} - Z_\alpha$$
  - Normalized error $$\varepsilon^R_\alpha = \frac{Z^*_{\alpha_0} - Z_\alpha}{\sigma_\alpha}$$

o Principle

- Kriging considers available samples in the system
- When too many samples, kriging system becomes very large and may become difficult to invert (unstable, slow)
- Kriging weights of peripheral points are small: could they be neglected?

Neighborhood:

- *Unique*: Take all data available
- *Moving*: Select the most appropriate subset of neighboring samples
  - By number
  - By maximum distance
  - By angular sector